CS255: Intro. to Cryptography **Winter 2023**

Assignment #3

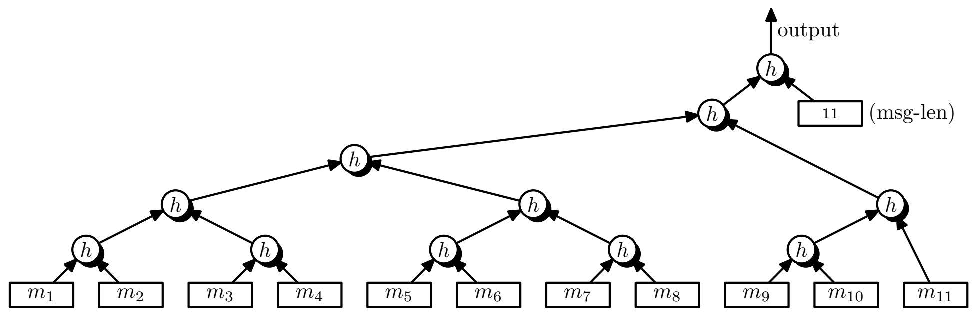
Due: Tuesday, Feb. 21, 2022, by Gradescope (each answer on a separate page).

Problem 1. RawCBC attacks. In class we discussed the ECBC (encrypted CBC) MAC for messages in $\mathcal{X}^{\leq L}$ where $\mathcal{X} = \{0,1\}^n$. Recall that RawCBC is the same as ECBC, but without the very last encryption step. We showed that RawCBC is an insecure MAC for variable length messages. Here we show a more devastating attack on RawCBC. Let m_1 and m_2 be two multi-block messages. Show that by asking the signer for the MAC tag on m_1 and for the MAC tag on one additional multi-block message m_2' of the same length as m_2, the attacker can obtain the MAC tag on $m = m_1 \parallel m_2$, the concatenation of m_1 and m_2.

Problem 2. Multicast MACs. Suppose user A wants to broadcast a message to n recipients $B_1, \ldots, B_n$. Privacy is not important but integrity is. In other words, each of $B_1, \ldots, B_n$ should be assured that the message he is receiving were sent by A. User A decides to use a MAC.

- **a.** Suppose user A and $B_1, \ldots, B_n$ all share a secret key k. User A computes the MAC tag for every message she sends using k. Every user B_i verifies the tag using k. Using at most two sentences explain why this scheme is insecure, namely, show that user B_1 is not assured that messages he is receiving are from A.
- **b.** Suppose user A has a set $S = \{k_1, \ldots, k_\ell\}$ of ℓ secret keys. Each user B_i has some subset $S_i \subseteq S$ of the keys. When A transmits a message she appends ℓ MAC tags to it by MACing the message with each of her ℓ keys. When user B_i receives a message he accepts it as valid only if all tags corresponding to keys in S_i are valid. Let us assume that the users $B_1, \ldots, B_n$ do not collude with each other. What property should the sets $S_1, \ldots, S_n$ satisfy so that the attack from part (a) does not apply?
- **c.** Show that when $n = 10$ (i.e. ten recipients) it suffices to take $\ell = 5$ in part (b). Describe the sets $S_1, \ldots, S_{10} \subseteq \{k_1, \ldots, k_5\}$ you would use.
- **d.** Show that the scheme from part (c) is completely insecure if two users are allowed to collude.

Problem 3. Parallel Merkle-Damgård. Recall that the Merkle-Damgård construction gives a *sequential* method for extending the domain of a CRHF. The tree construction in the figure below is a parallelizable approach: all the hash functions h within a single level can be computed in parallel. Prove that the resulting hash function defined over $(\mathcal{X}^{\leq L},\ \mathcal{X})$ is collision resistant, assuming h is collision resistant. Here h is a compression function $h : \mathcal{X}^2 \to \mathcal{X}$, and we assume the message length can be encoded as an element of $\mathcal{X}$.



More precisely, the hash function is defined as follows:

input: $m_1 \dots m_s \in \mathcal{X}^s$ for some $1 \leq s \leq L$
output: $y \in \mathcal{X}$

let $t \in \mathbb{Z}$ be the smallest power of two such that $t \geq s$ (i.e., $t := 2^{\lceil \log_2 s \rceil}$)
for $i = s+1$ to t: $m_i \leftarrow \perp$
for $i = t+1$ to $2t-1$:
  $\ell \leftarrow 2(i-t)-1,\ \ r \leftarrow \ell + 1$ // *indices of left and right children*
  if $m_\ell = \perp$ and $m_r = \perp$: $m_i \leftarrow \perp$ // *if node has no children, set node to null*
  else if $m_r = \perp$: $m_i \leftarrow m_\ell$ // *if one child, propagate child as is*
  else $m_i \leftarrow h(m_\ell, m_r)$ // *if two children, hash with h*
output $y \leftarrow h\big(m_{2t-1},\ s\big)$ // *hash final output and message length*

Problem 4. In the lecture we saw that Davies-Meyer is used to convert an ideal block cipher into a collision resistant compression function. Let $E(k, m)$ be a block cipher where the message space is the same as the key space (e.g. 128-bit AES). Show that the following methods do not work:

$$f_1(x, y) = E(y, x) \oplus y \quad \text{and} \quad f_2(x, y) = E(x,\ x \oplus y)$$

That is, show an efficient algorithm for constructing collisions for f_1 and f_2. Recall that the block cipher E and the corresponding decryption algorithm D are both known to you.

Problem 5. Authenticated encryption. Let (E, D) be an encryption system that provides authenticated encryption. Here E does not take a nonce as input and therefore must be a randomized encryption algorithm. Which of the following systems provide authenticated encryption? For those that do, give a short proof. For those that do not, present an attack that either breaks CPA security or ciphertext integrity.

a. $E_1(k,m) = \big[c \leftarrow E(k,m),\ \text{output}\ \ (c,c)\ \big]$ and $D_1(k,\ (c_1,c_2)\ ) = D(k,c_1)$

b. $E_2(k,m) = \big[c \leftarrow E(k,m),\ \text{output}\ \ (c,c)\ \big]$ and $D_2(k,\ (c_1,c_2)\ ) = \begin{cases} D(k,c_1) & \text{if } c_1 = c_2 \\ \mathsf{fail} & \text{otherwise} \end{cases}$

c. $E_3(k,m) = \big(\ E(k,m),\ E(k,m)\ \big)$ and $D_3(k,\ (c_1,c_2)\ ) = \begin{cases} D(k,c_1) & \text{if } D(k,c_1) = D(k,c_2) \\ \mathsf{fail} & \text{otherwise} \end{cases}$

To clarify: $E(k,m)$ is randomized so that running it twice on the same input will result in different outputs with high probability.

d. $E_4(k,m) = \big(\ E(k,m),\ H(m)\ \big)$ and $D_4(k,\ (c_1,c_2)\ ) = \begin{cases} D(k,c_1) & \text{if } H(D(k,c_1)) = c_2 \\ \mathsf{fail} & \text{otherwise} \end{cases}$

where H is a collision resistant hash function.

Problem 6. Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$ where $\mathcal{Y} := \{0,1\}^n$. Let $(E_{\text{ctr}}, D_{\text{ctr}})$ be the cipher derived from F using randomized counter mode. Let $H : \mathcal{Y}^{\leq L} \to \mathcal{Y}$ be a collision resistant hash function. Consider the following attempt at building an AE-secure cipher defined over $(\mathcal{K},\ \mathcal{Y}^{\leq L}, \mathcal{Y}^{\leq L+2})$:

$$E'(k,m) := E_{\text{ctr}}\big(k,\ (H(m), m)\big)\ ; \qquad D'(k,c) := \left\{ \begin{array}{l} (t,m) \leftarrow D_{\text{ctr}}(k,c) \\ \text{if } t = H(m) \text{ output } m, \text{ else } \mathsf{reject} \end{array} \right\}$$

Note that when encrypting a single block message $m \in \mathcal{Y}$, the output is three blocks: the random IV, a ciphertext block corresponding to $H(m)$, and a ciphertext block corresponding to m. Show that (E', D') is not AE-secure by showing that it does not have ciphertext integrity. Your attack should make a single encryption query.

At some point in the past, this type of construction was used to protect secret keys in the Android KeyStore. Your attack resulted in a compromise of the key store.

Problem 7. Alice and Bob run the Diffie-Hellman protocol in the cyclic group $\mathbb{G} = \mathbb{Z}_{101}^*$ with generator $g = 7$. What is the Diffie-Hellman secret $s = g^{ab} \in \mathbb{G}$ if Alice uses $a = 3$ and Bob uses $b = 67$? You do not need a calculator to solve this problem!

Problem 8. Exponentiation algorithms. Let $\mathbb{G}$ be a finite cyclic group of order p with generator g. In class we discussed the repeated squaring algorithm for computing $g^x \in \mathbb{G}$ for $0 \leq x < p$. The algorithm needed at most $2\log_2 p$ multiplications in $\mathbb{G}$.

In this question we develop a faster exponentiation algorithm. For some small constant w, called the window size, the algorithm begins by building a table T of size 2^w defined as follows:

$$\text{set}\ \ T[k] := g^k \ \text{ for } k = 0, \ldots, 2^w - 1\ . \tag{1}$$

a. Show that once the table T is computed, we can compute g^x using only $(1+1/w)(\log_2 p)$ multiplications in $\mathbb{G}$. Your algorithm shows that when the base of the exponentiation g is fixed forever, the table T can be pre-computed once and for all. Then exponentiation is faster than with repeated squaring.
Hint: Start by writing the exponent x base 2^w so that:

$$x = x_0 + x_1 2^w + x_2 (2^w)^2 + \ldots + x_{d-1}(2^w)^{d-1} \quad \text{where } 0 \leq x_i < 2^w \text{ for all } i = 0, \ldots, d-1.$$

Here there are d digits in the representation of x base 2^w. Start the exponentiation algorithm with x_{d-1} and work your way down, squaring the accumulator w times at every iteration.

b. Suppose every exponentiation is done relative to a different base, so that a new table T must be re-computed for every exponentiation. What is the worse case number of multiplications as a function of w and $\log_2 p$?

c. Continuing with Part (b), compute the optimal window size w when $\log_2 p = 256$, namely the w that minimizes the overall worst-case running time. What is the worst-case running time with this w? (counting only multiplications in $\mathbb{G}$)