

CS255: Winter 2019

PRPs and PRFs

1. Abstract block ciphers: PRPs and PRFs,
2. Security models for encryption,
3. Analysis of CBC and counter mode

PRPs and PRFs

- Pseudo Random Function (**PRF**) defined over (K, X, Y) :

$$F: K \times X \rightarrow Y$$

such that exists “efficient” algorithm to evaluate $F(k, x)$

- Pseudo Random Permutation (**PRP**) defined over (K, X) :

$$E: K \times X \rightarrow X$$

such that:

1. Exists “efficient” algorithm to evaluate $E(k, x)$
2. The function $E(k, \cdot)$ is one-to-one
3. Exists “efficient” inversion algorithm $D(k, x)$

Running example

- Example PRPs: 3DES, AES, ...

AES-128: $K \times X \rightarrow X$ where $K = X = \{0,1\}^{128}$

DES: $K \times X \rightarrow X$ where $X = \{0,1\}^{64}$, $K = \{0,1\}^{56}$

3DES: $K \times X \rightarrow X$ where $X = \{0,1\}^{64}$, $K = \{0,1\}^{168}$

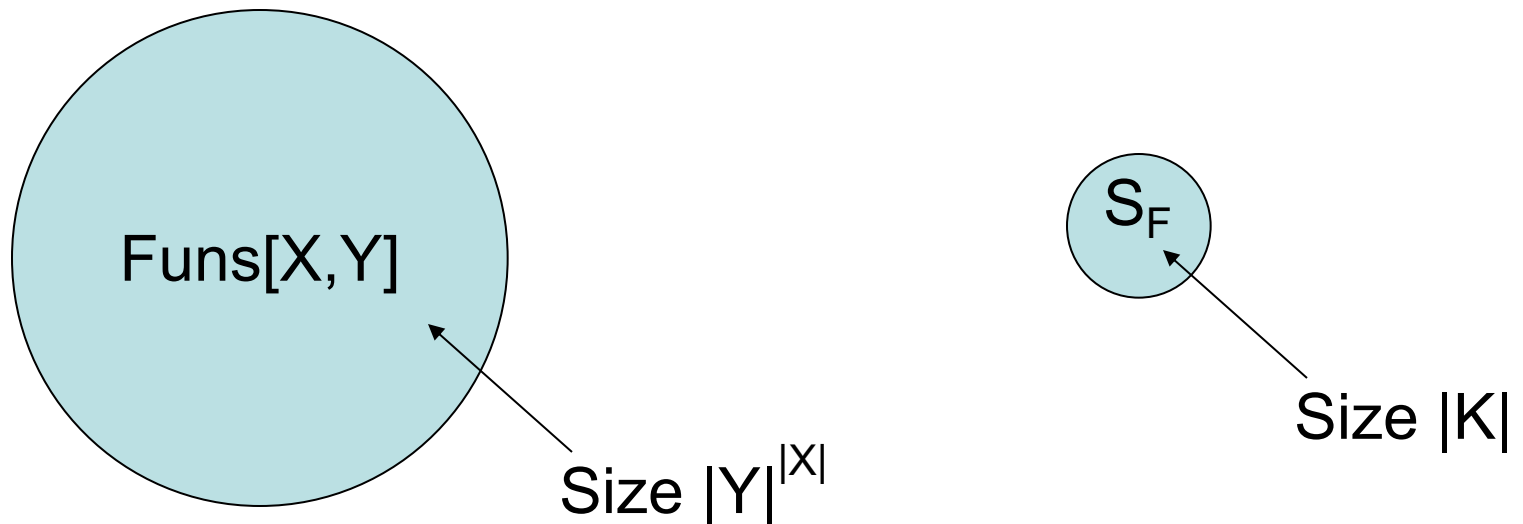
- Functionally, any PRP is also a PRF.
 - A PRP is a PRF where $X=Y$ and is efficiently invertible
 - A PRP is sometimes called a ***block cipher***

Secure PRFs

- Let $F: K \times X \rightarrow Y$ be a PRF

$$\left\{ \begin{array}{l} \text{Funs}[X,Y]: \text{ the set of all functions from } X \text{ to } Y \\ S_F = \{ F(k, \cdot) \text{ s.t. } k \in K \} \subseteq \text{Funs}[X,Y] \end{array} \right.$$

- Intuition: a PRF is **secure** if
a random function in $\text{Funs}[X,Y]$ is indistinguishable from
a random function in S_F

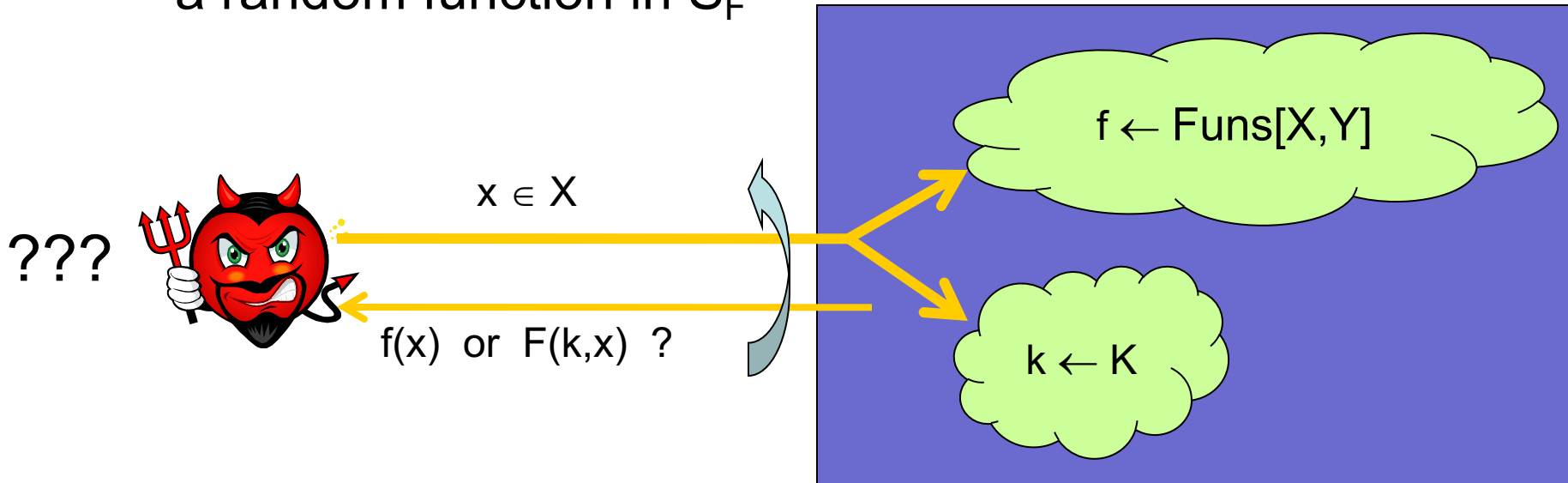


Secure PRFs

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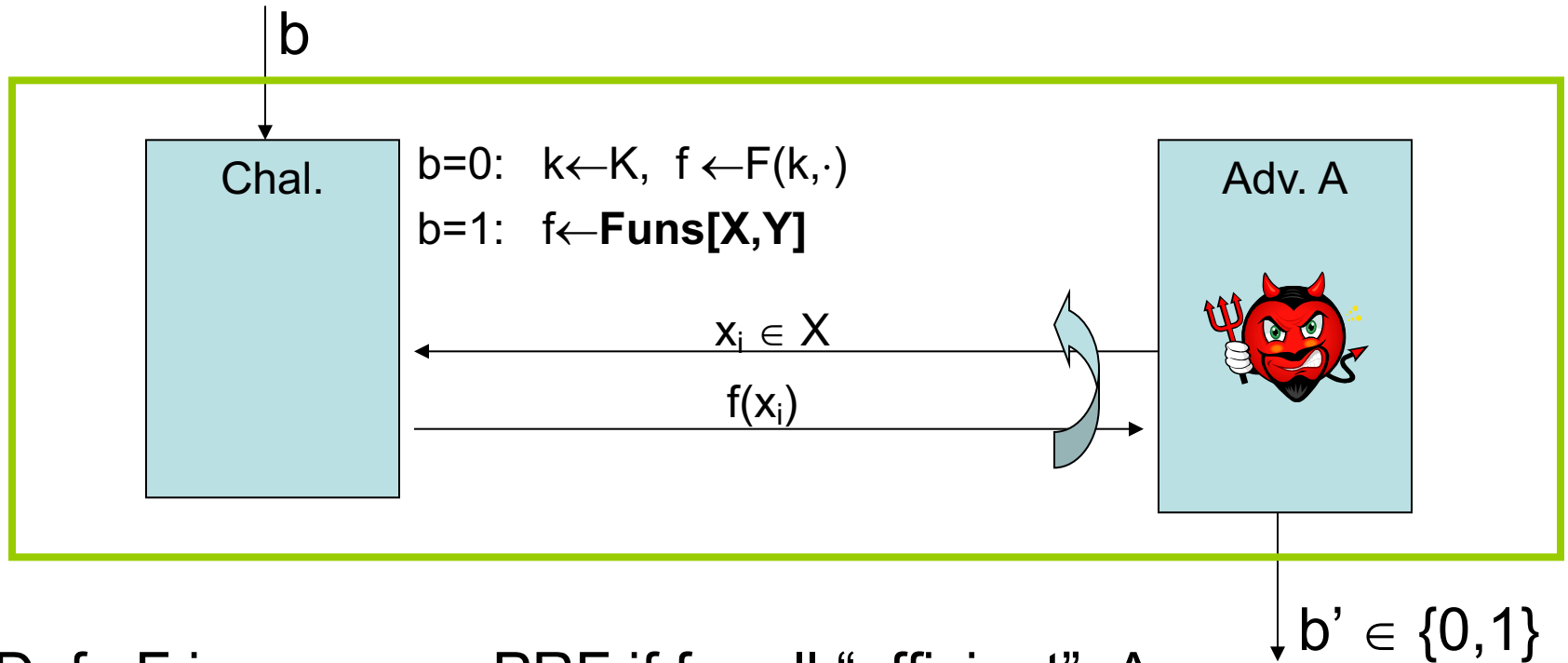
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- Intuition: a PRF is **secure** if a random function in $\text{Funs}[X,Y]$ is indistinguishable from a random function in S_F



Secure PRF: definition

- For $b=0,1$ define experiment $\text{EXP}(b)$ as:



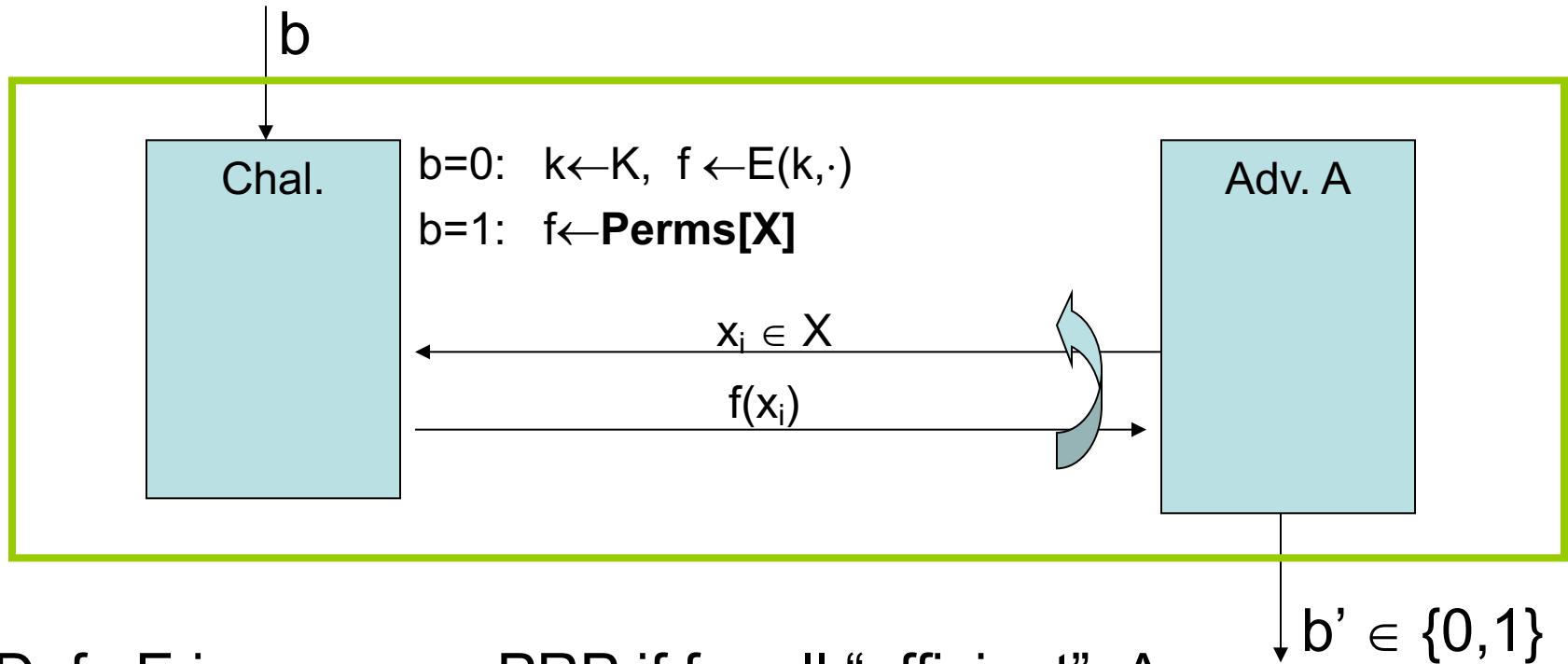
- Def: F is a secure PRF if for all “efficient” A :

$$\text{Adv}_{\text{PRF}}[A, F] = \left| \Pr[\text{EXP}(0)=1] - \Pr[\text{EXP}(1)=1] \right|$$

is “negligible.”

Secure PRP

- For $b=0,1$ define experiment $\text{EXP}(b)$ as:



- Def: E is a secure PRP if for all “efficient” A :

$$\text{Adv}_{\text{PRP}}[A, E] = \left| \Pr[\text{EXP}(0)=1] - \Pr[\text{EXP}(1)=1] \right|$$

is “negligible.”

Example secure PRPs

- Example secure PRPs: 3DES, AES, ...

$\text{AES}_{256}: K \times X \rightarrow X$ where $X = \{0,1\}^{128}$

$K = \{0,1\}^{256}$

- AES_{256} PRP Assumption (example) :

All explicit 2^{80} -time algs A have $\text{PRP Adv}[A, \text{AES}_{256}] < 2^{-40}$

PRF Switching Lemma

Any secure PRP is also a secure PRF.

Lemma: Let E be a PRP over (K, X)

Then for any q -query adversary A :

$$\left| \text{Adv}_{\text{PRF}}[A, E] - \text{Adv}_{\text{PRP}}[A, E] \right| < q^2 / 2|X|$$

\Rightarrow Suppose $|X|$ is large so that $q^2 / 2|X|$ is “negligible”

Then $\text{Adv}_{\text{PRP}}[A, E]$ “negligible” $\Rightarrow \text{Adv}_{\text{PRF}}[A, E]$ “negligible”

Using PRPs and PRFs

- Goal: build “secure” encryption from a PRP.
- Security is always defined using two parameters:

1. What “**power**” does adversary have?

examples:

- Adv sees only one ciphertext (one-time key)
- Adv sees many PT/CT pairs (many-time key, CPA)

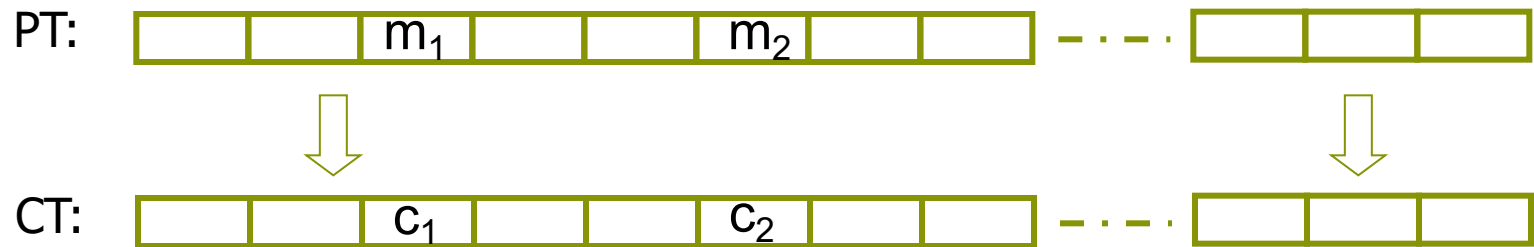
2. What “**goal**” is adversary trying to achieve?

examples:

- Fully decrypt a challenge ciphertext.
- Learn info about PT from CT (semantic security)

Incorrect use of a PRP

Electronic Code Book (ECB):



Problem:

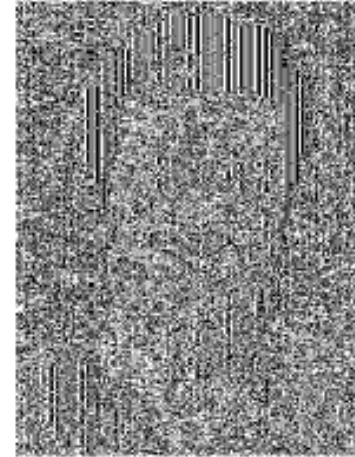
– if $m_1 = m_2$ then $c_1 = c_2$

In pictures

An example plaintext



Encrypted with AES in ECB mode



(courtesy B. Preneel)

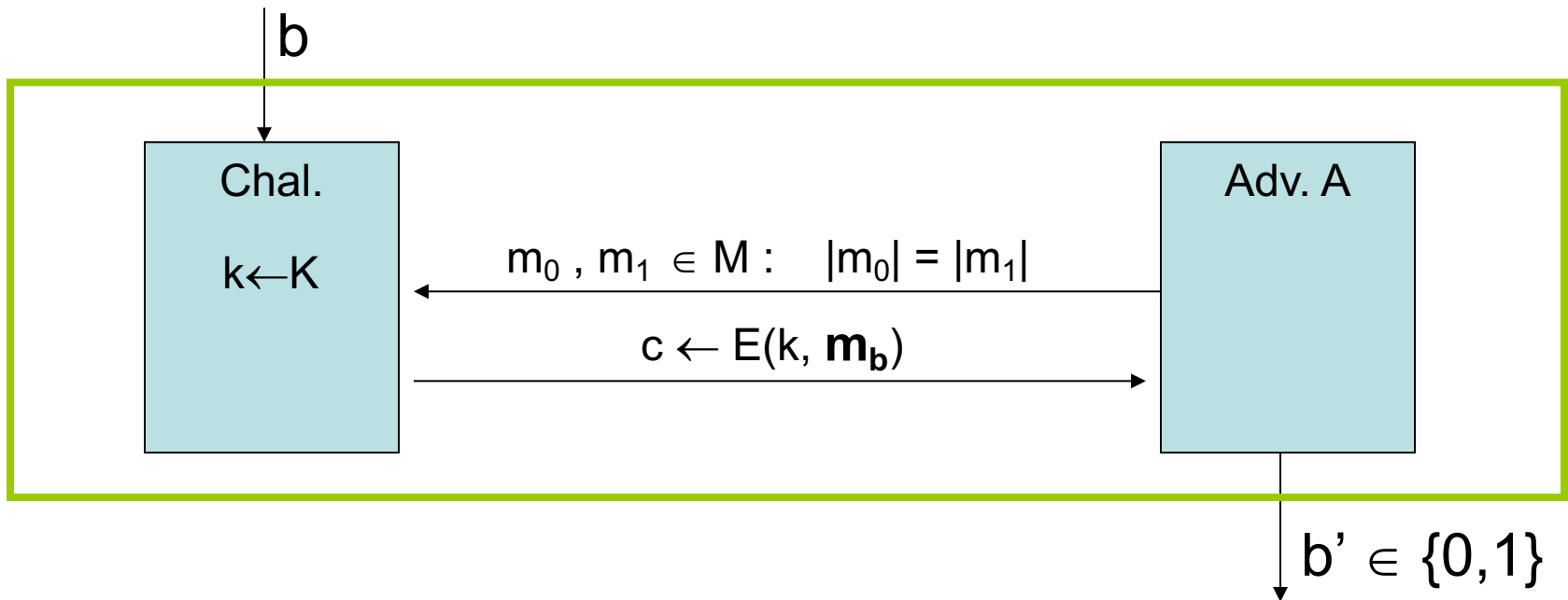
Modes of Operation for One-time Use Key

Example application:

Encrypted email. New key for every message.

Semantic Security for one-time key

- $\mathbb{E} = (E,D)$ a cipher defined over (K,M,C)
- For $b=0,1$ define $\text{EXP}(b)$ as:



- Def: \mathbb{E} is sem. sec. for one-time key if for all “efficient” A :

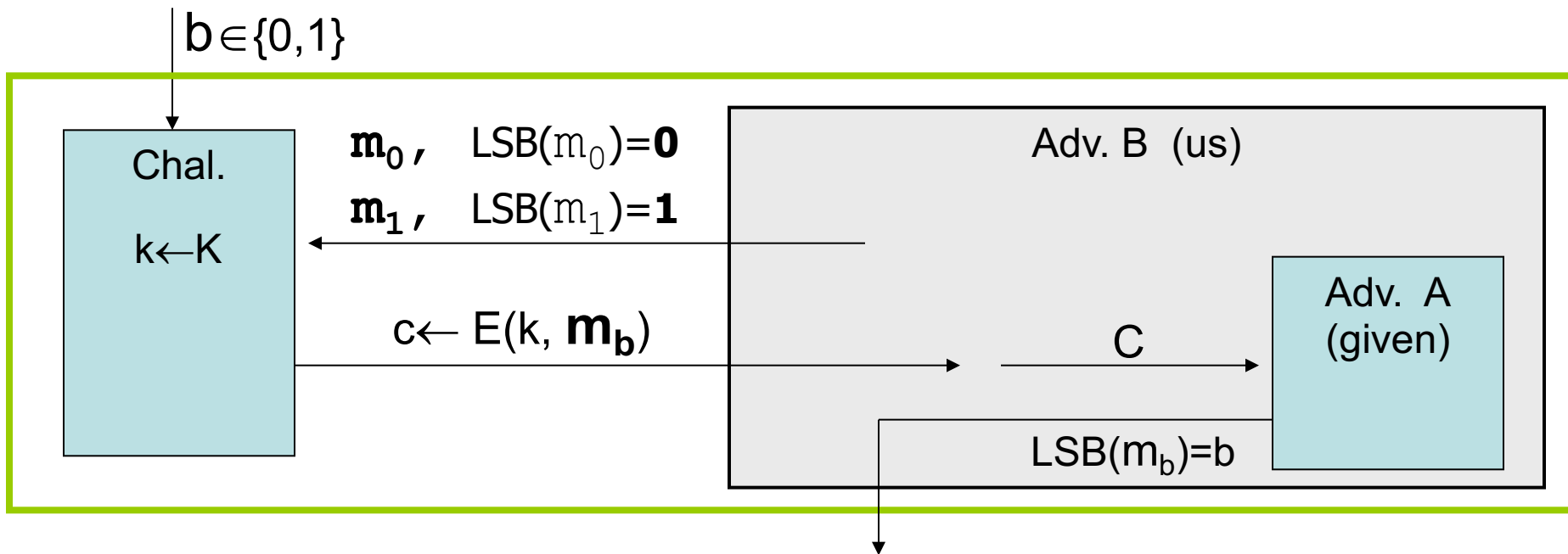
$$\text{Adv}_{\text{SS}}[A, \mathbb{E}] = \left| \Pr[\text{EXP}(0)=1] - \Pr[\text{EXP}(1)=1] \right|$$

is “negligible.”

Semantic security (cont.)

Sem. Sec. \Rightarrow no “efficient” adversary learns info about PT from a **single** CT.

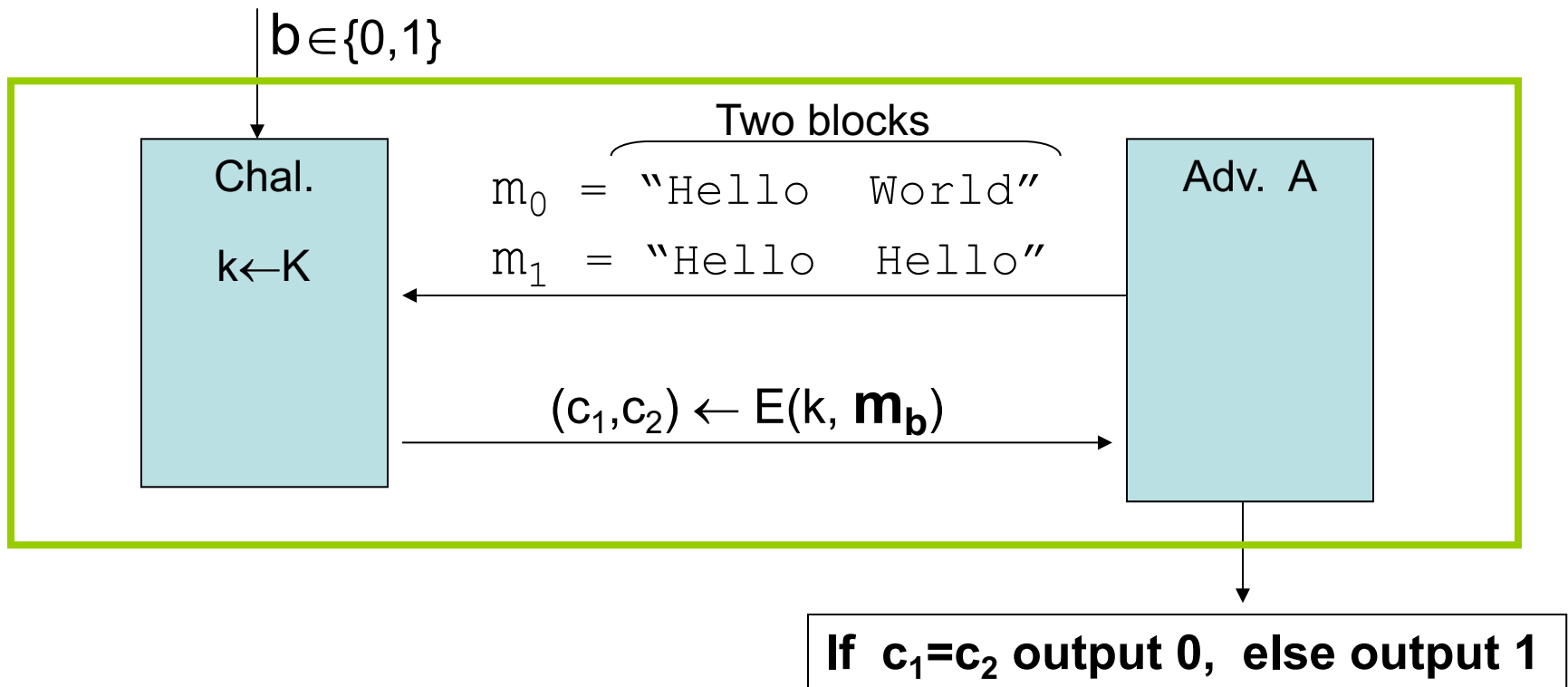
Example: suppose efficient A can deduce LSB of PT from CT. Then $\mathbb{E} = (E, D)$ is not semantically secure.



Then $\text{Adv}_{\text{SS}}[B, \mathbb{E}] = 1 \Rightarrow \mathbb{E}$ is not sem. sec.

Note: ECB is not Sem. Sec.

ECB is not semantically secure for messages that contain two or more blocks.



Then $\text{Adv}_{\text{SS}}[A, \text{ECB}] = 1$

Secure Constructions

Examples of sem. sec. systems:

1. $\text{Adv}_{\text{SS}}[A, \text{OTP}] = 0$ for all A

2. Deterministic counter mode from a PRF F :

• $E_{\text{DETCTR}}(k,m) =$

$$\begin{array}{cccc} m[0] & m[1] & \dots & m[L] \\ \oplus & & & \\ F(k,0) & F(k,1) & \dots & F(k,L) \\ \hline c[0] & c[1] & \dots & c[L] \end{array}$$

• Stream cipher built from PRF (e.g. AES, 3DES)

Det. counter-mode security

Theorem: For any $L > 0$.

If F is a secure PRF over (K, X, X) then

E_{DETCTR} is sem. sec. cipher over (K, X^L, X^L) .

In particular, for any adversary A attacking E_{DETCTR} there exists a PRF adversary B s.t.:

$$\text{Adv}_{\text{SS}}[A, E_{\text{DETCTR}}] = 2 \cdot \text{Adv}_{\text{PRF}}[B, F]$$

$\text{Adv}_{\text{PRF}}[B, F]$ is negligible (since F is a secure PRF)

$\Rightarrow \text{Adv}_{\text{SS}}[A, E_{\text{DETCTR}}]$ must be negligible.

Modes of Operation for Many-time Key

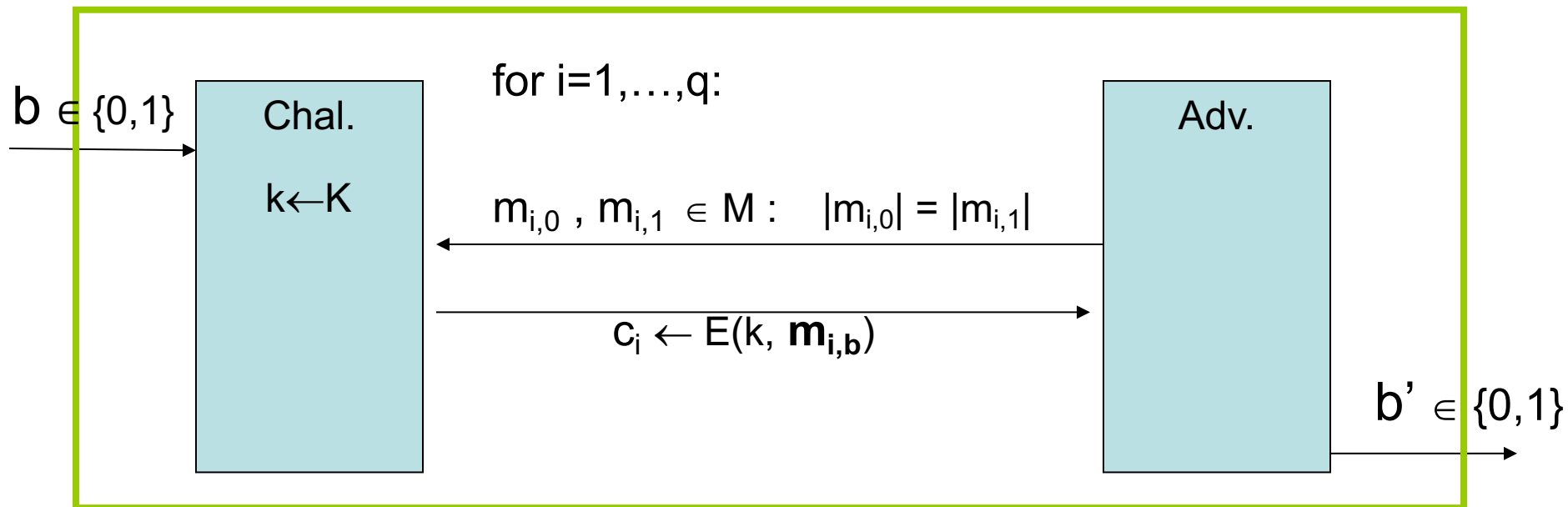
Example applications:

1. File systems: Same AES key used to encrypt many files.
2. IPsec: Same AES key used to encrypt many packets.

Semantic Security for many-time key (CPA security)

Cipher $\mathbb{E} = (E, D)$ defined over (K, M, C) .

For $b=0,1$ define $\text{EXP}(b)$ as:



if adv. wants $c = E(k, m)$ it queries with $m_{j,0} = m_{j,1} = m$

Def: \mathbb{E} is sem. sec. under CPA if for all “efficient” A :

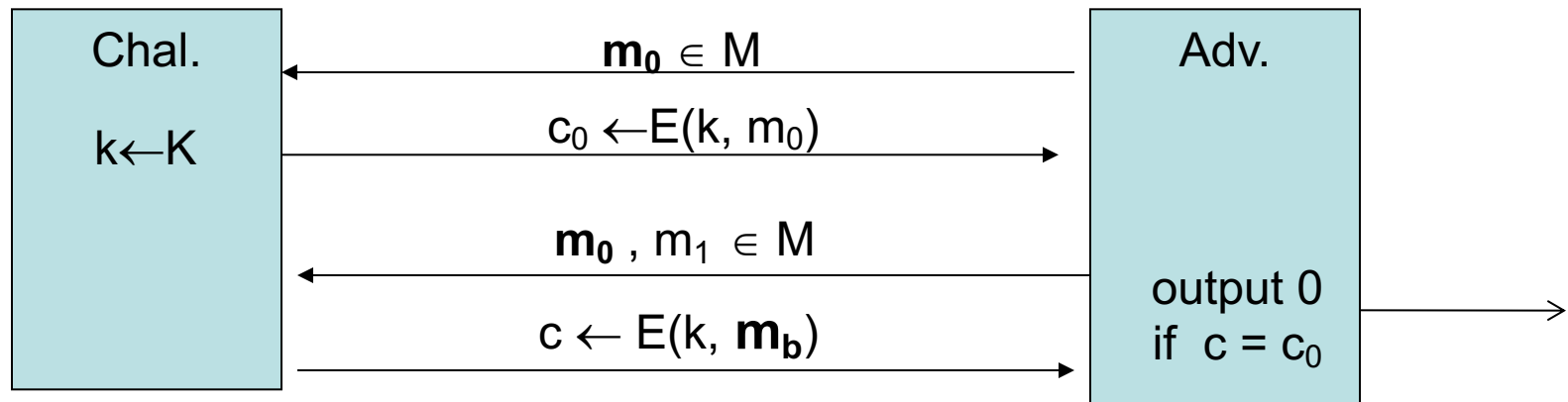
$$\text{Adv}_{\text{CPA}} [A, \mathbb{E}] = \left| \Pr[\text{EXP}(0)=1] - \Pr[\text{EXP}(1)=1] \right|$$

is “negligible.”

Security for many-time key

Fact: stream ciphers are insecure under CPA.

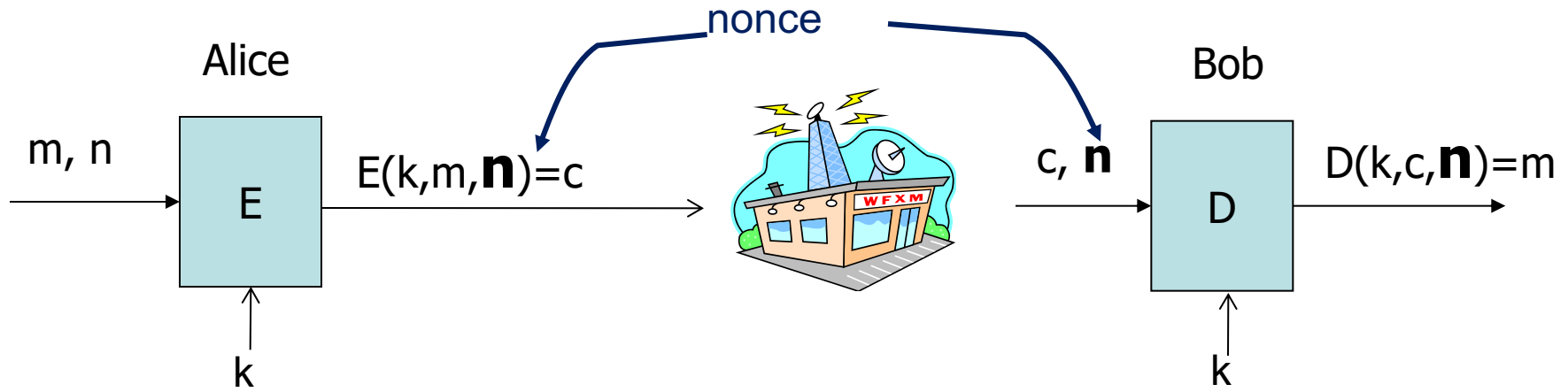
- More generally: if $E(k,m)$ always produces same ciphertext, then cipher is insecure under CPA.



If secret key is to be used multiple times \Rightarrow

given the same plaintext message twice,
the encryption alg. must produce different outputs.

Nonce-based Encryption

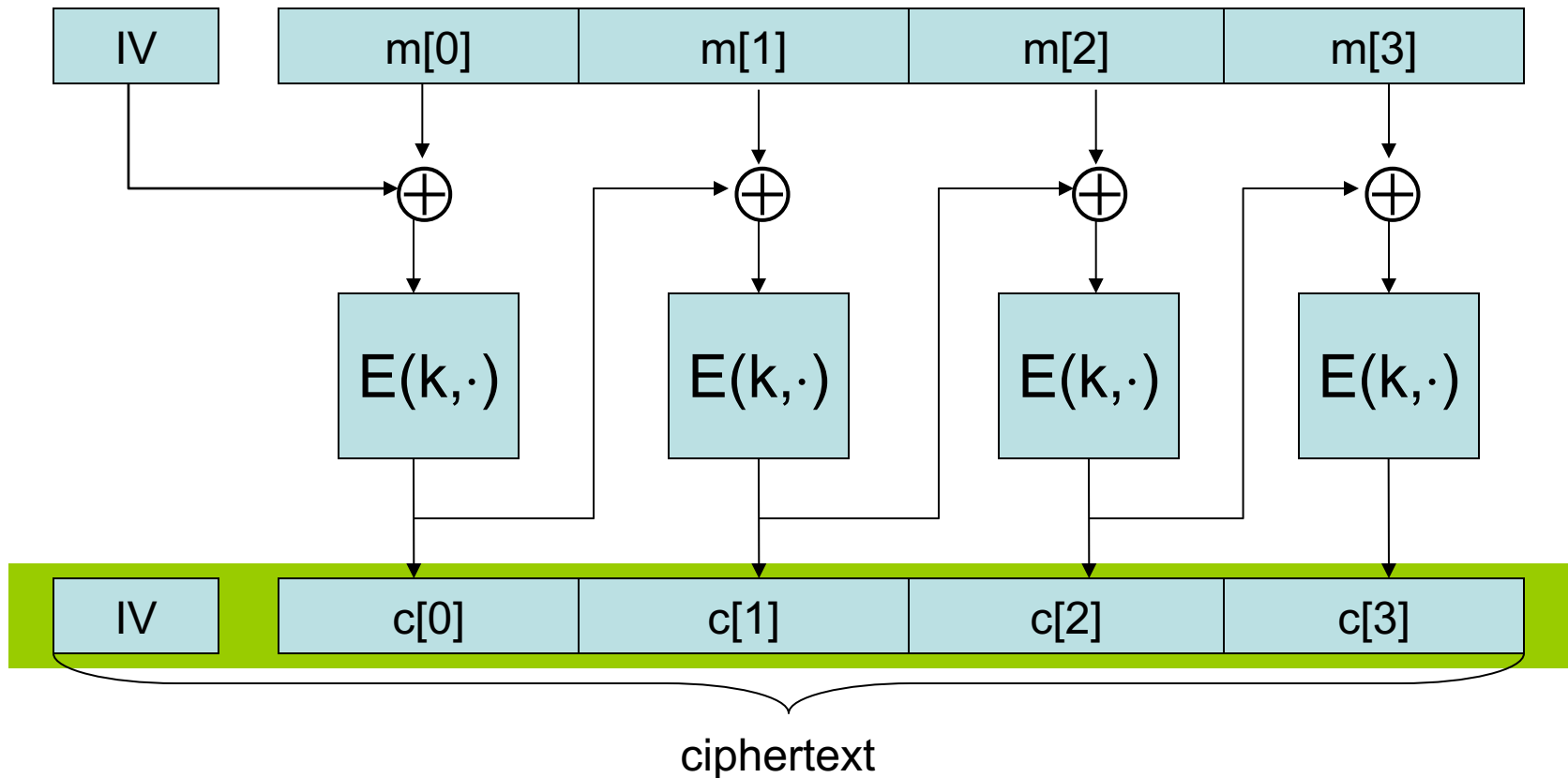


nonce n : a value that changes from msg to msg
(k, n) pair never used more than once

- method 1: encryptor chooses a random nonce, $n \leftarrow \mathcal{N}$
- method 2: nonce is a counter (e.g. packet counter)
 - used when encryptor keeps state from msg to msg
 - if decryptor has same state, need not send nonce with CT

Construction 1: CBC with random nonce

Cipher block chaining with a random IV (IV = nonce)



CBC: CPA Analysis

CBC Theorem: For any $L > 0$,

If E is a secure PRP over (K, X) then

E_{CBC} is a sem. sec. under CPA over (K, X^L, X^{L+1}) .

In particular, for a q -query adversary A attacking E_{CBC} there exists a PRP adversary B s.t.:

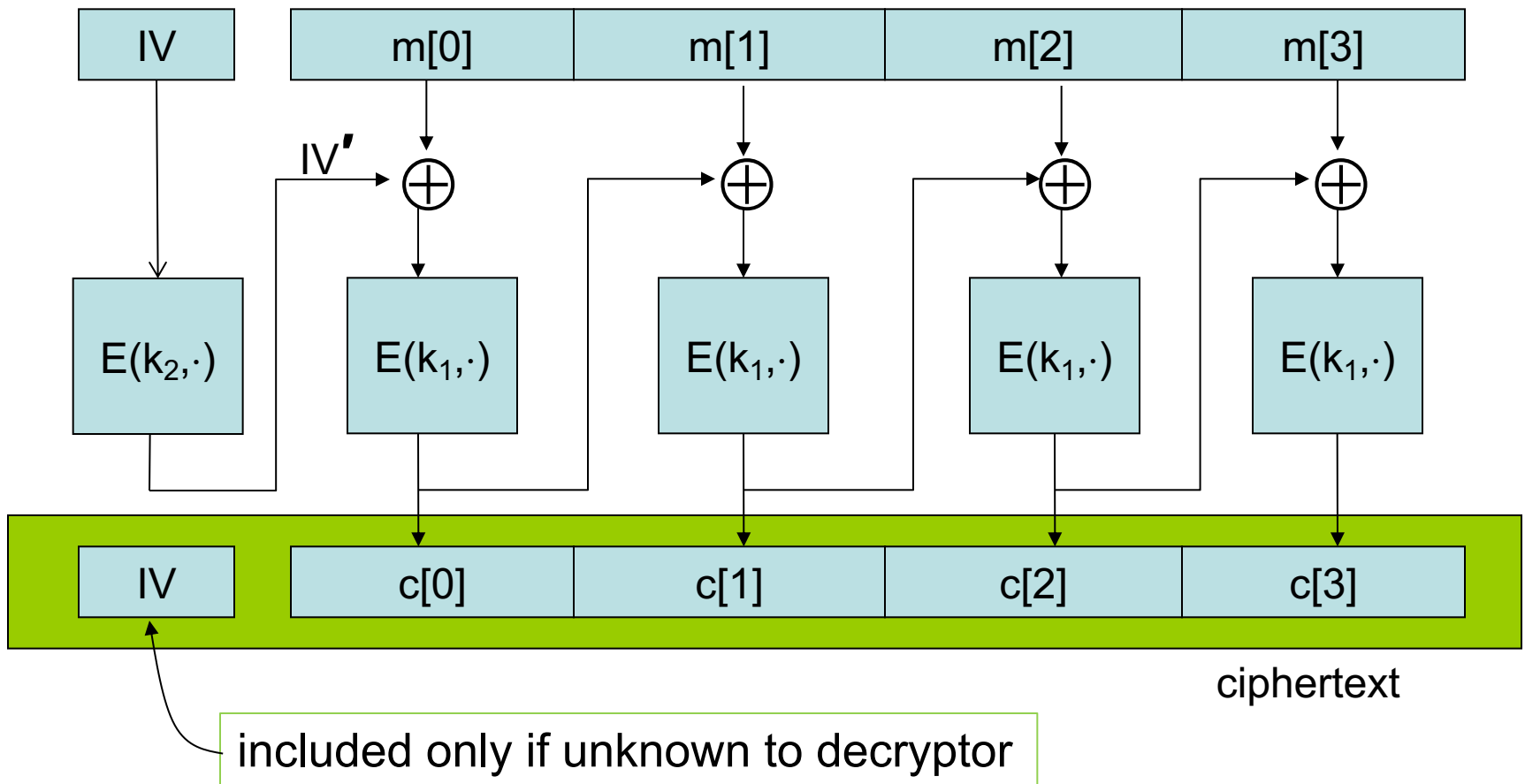
$$\text{Adv}_{\text{CPA}}[A, E_{\text{CBC}}] \leq 2 \cdot \text{Adv}_{\text{PRP}}[B, E] + 2q^2 L^2 / |X|$$

Note: CBC is only secure as long as $q^2 L^2 \ll |X|$

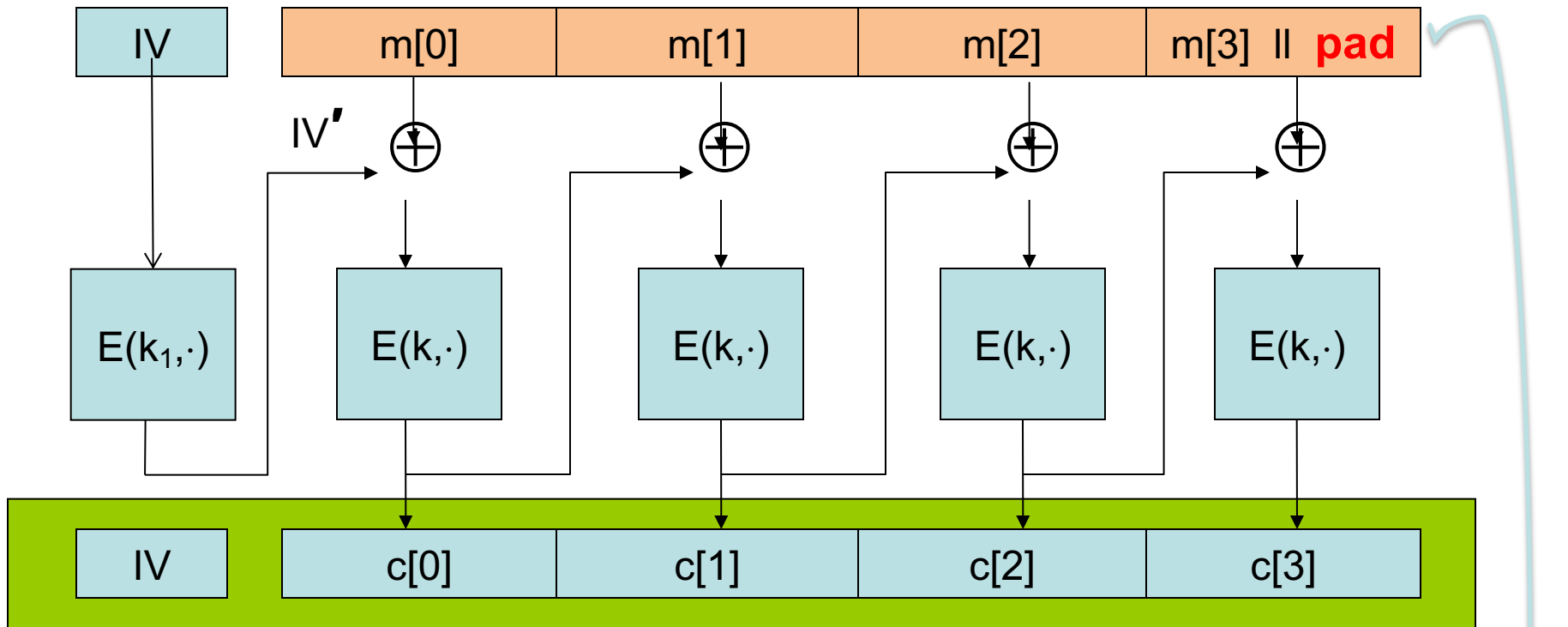
Construction 1': CBC with **unique** nonce

Cipher block chaining with unique IV (IV = nonce)

unique IV means: (key,IV) pair is used for only one message



A CBC technicality: padding



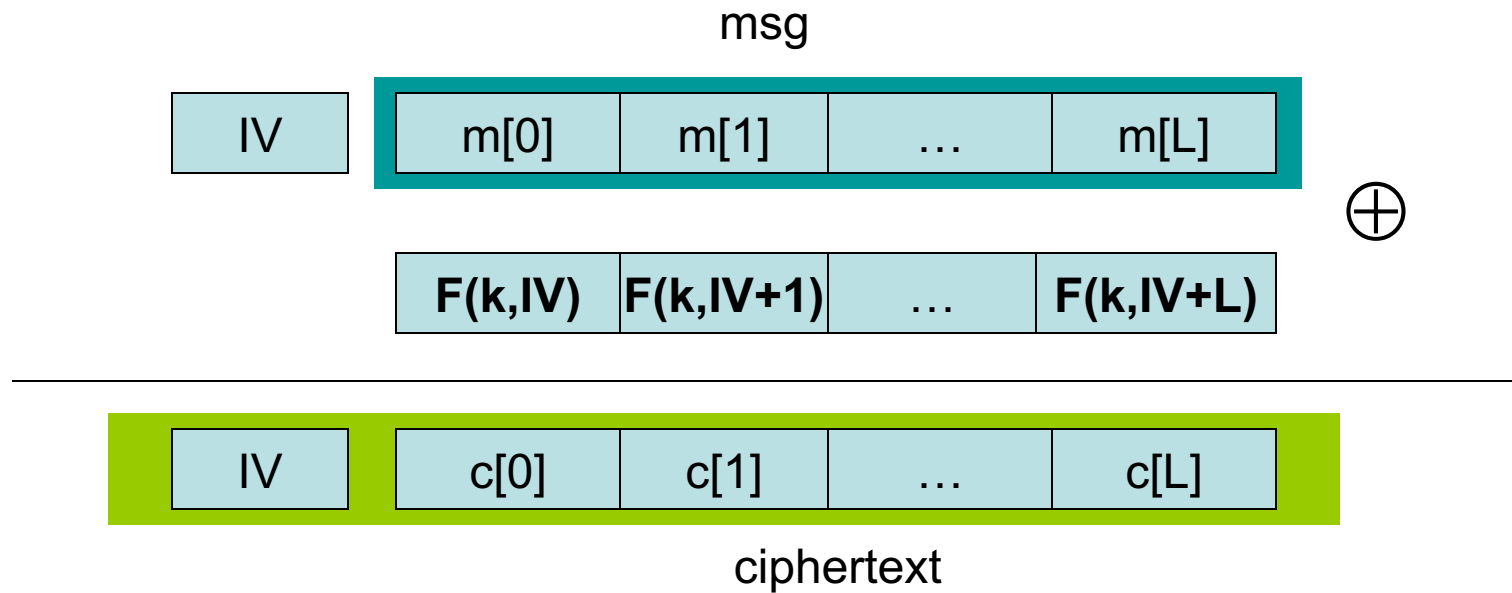
TLS 1.0: for $n > 0$, $n+1$ byte pad is

n	n	n	...	n
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if no pad needed, add a dummy block

removed during decryption

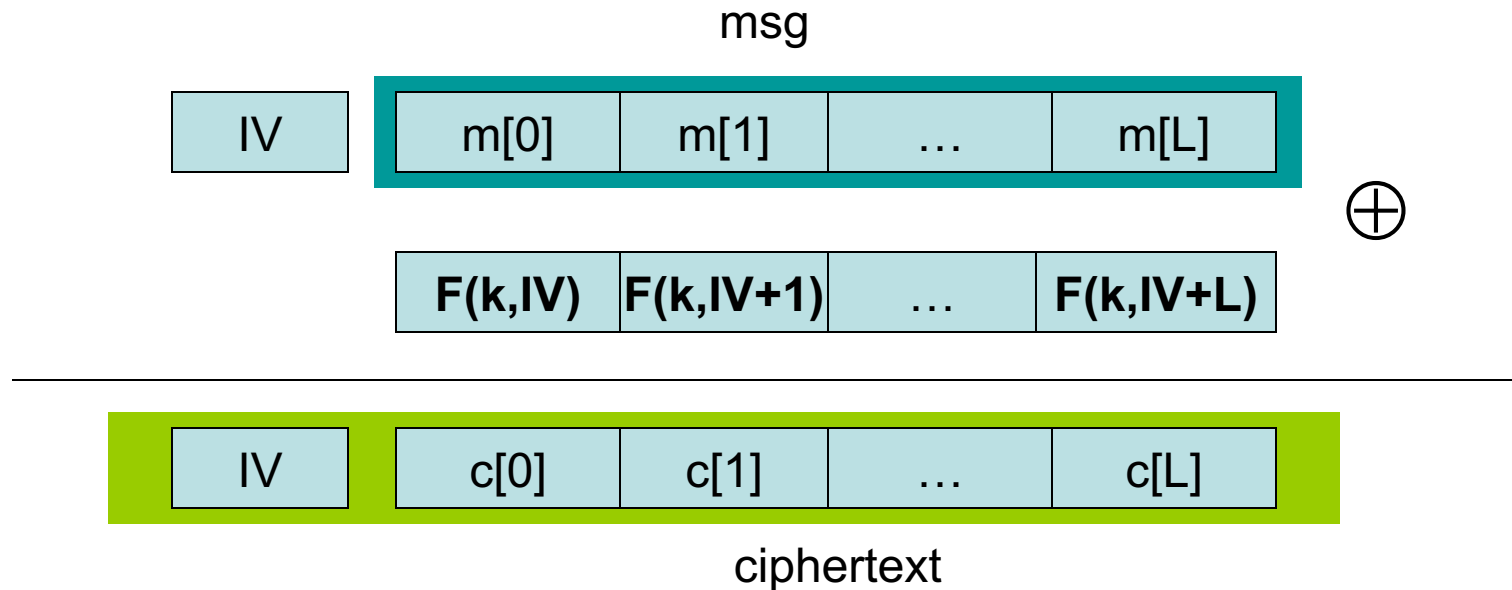
Construction 2: rand ctr-mode



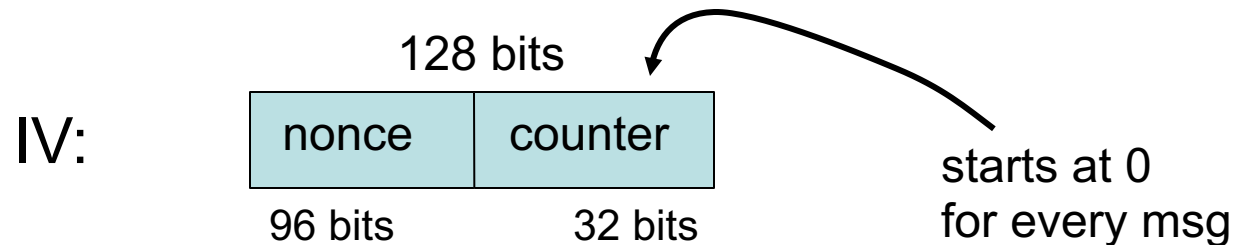
IV - chosen at random for every message

note: parallelizable (unlike CBC)

Construction 2': nonce ctr-mode



To ensure $F(K, x)$ is never used more than once, choose IV as:



rand ctr-mode: CPA analysis

Randomized counter mode: random IV.

Counter-mode Theorem: For any $L > 0$,

If F is a secure PRF over (K, X, X) then

E_{CTR} is a sem. sec. under CPA over (K, X^L, X^{L+1}) .

In particular, for a q -query adversary A attacking E_{CTR} there exists a PRF adversary B s.t.:

$$\text{Adv}_{CPA}[A, E_{CTR}] \leq 2 \cdot \text{Adv}_{PRF}[B, F] + 2q^2L / |X|$$

Note: ctr-mode only secure as long as $q^2L \ll |X|$

Better than CBC !

An example

$$\text{Adv}_{\text{CPA}}[A, E_{\text{CTR}}] \leq 2 \cdot \text{Adv}_{\text{PRF}}[B, E] + 2 q^2 L / |X|$$

q = # messages encrypted with k , L = length of max msg

Suppose we want $\text{Adv}_{\text{CPA}}[A, E_{\text{CTR}}] \leq q^2 L / |X| \leq 1 / 2^{32}$

- AES: $|X| = 2^{128} \Rightarrow q L^{1/2} < 2^{48}$

So, after 2^{32} CTs each of len 2^{32} , must change key
(total of 2^{64} AES blocks)

Comparison: ctr vs. CBC

	CBC	ctr mode
uses	PRP	PRF
parallel processing	No	Yes
Security of rand. enc.	$q^2 L^2 \ll X $	$q^2 L \ll X $
dummy padding block	Yes*	No
1 byte msgs (nonce-based)	16x expansion	no expansion

* for CBC, dummy padding block can be avoided using *ciphertext stealing*

Summary

PRPs and PRFs: a useful abstraction of block ciphers.

We examined two security notions:

1. Semantic security against one-time CPA.
2. Semantic security against many-time CPA.

Note: neither mode ensures data integrity.

Stated security results summarized in the following table:

Power Goal	one-time key	Many-time key (CPA)	CPA and CT integrity
Sem. Sec.	stream-ciphers det. ctr-mode	rand CBC rand ctr-mode	later