CS255: Cryptography and Computer Security **Winter 2017**

Assignment #3

Due: Monday, Mar. 13, 2017, by Gradescope (each answer on a seperate page).

Problem 1. Let's explore why in the RSA public key system each person has to be assigned a different modulus $n = pq$. Suppose we try to use the same modulus $n = pq$ for everyone. Each person is assigned a public exponent e_i and a private exponent d_i such that $e_i \cdot d_i = 1 \bmod \varphi(n)$. At first this appears to work fine: to encrypt to Bob, Alice computes $c = x^{e_{\text{bob}}}$ for some value x and sends c to Bob. An eavesdropper Eve, not knowing d_{bob} appears to be unable to invert Bob's RSA function to decrypt c. Let's show that using e_{eve} and d_{eve} Eve can very easily decrypt c.

a. Show that given e_{eve} and d_{eve} Eve can obtain a multiple of $\varphi(n)$. Let us denote that integer by V.

b. Suppose Eve intercepts a ciphertext $c = x^{e_{\text{bob}}} \bmod n$. Show that Eve can use V to efficiently obtain x from c. In other words, Eve can invert Bob's RSA function.
Hint: First, suppose e_{bob} is relatively prime to V. Then Eve can find an integer d such that $d \cdot e_{\text{bob}} = 1 \bmod V$. Show that d can be used to efficiently compute x from c. Next, show how to make your algorithm work even if e_{bob} is not relatively prime to V.

Note: In fact, one can show that Eve can completely factor the global modulus n.

Problem 2. Time-space tradeoff. Let $f : X \to X$ be a one-way one-to-one function. Show that one can build a table T of size $2B$ eleents of X ($B \ll |X|$) that enables an attacker to invert f in time $O(|X|/B)$. More precisely, construct an $O(|X|/B)$-time deterministic algorithm $\mathcal{A}$ that takes as input the table T and a $y \in X$, and outputs an $x \in X$ satisfying $f(x) = y$. This result suggests that the more memory the attacker has, the easier it becomes to invert functions.
Hint: Pick a random point $z \in X$ and compute the sequence

$$z_0 := z, \;\; z_1 := f(z), \;\; z_2 := f(f(z)), \;\; z_3 := f(f(f(z))), \;\; \ldots$$

Since f is a permutation, this sequence must come back to z at some point (i.e. there exists some $j > 0$ such that $z_j = z$). We call the resulting sequence $(z_0, z_1, \ldots, z_j)$ an f-cycle. Let $t := \lceil |X|/B \rceil$. Try storing $(z_0, z_t, z_{2t}, z_{3t}, \ldots)$ in memory. Use this table (or perhaps, several such tables) to invert an input $y \in X$ in time $O(t)$.

Problem 3. Let's build a collision resistant hash function from the RSA problem. Let n be a random RSA modulus, e a prime relatively prime to $\varphi(n)$, and u random in $\mathbb{Z}_n^*$. Show that the function

$$H_{n,u,e} : \mathbb{Z}_n^* \times \{0, \ldots, e-1\} \to \mathbb{Z}_n^* \qquad \text{defined by} \qquad H_{n,u,e}(x, y) := x^e u^y \quad \in \mathbb{Z}_n \tag{1}$$

is collision resistant assuming that taking e'th roots modulo n is hard.

Suppose $\mathcal{A}$ is an algorithm that takes n, u as input and outputs a collision for $H_{n,u,e}(\cdot, \cdot)$. Your goal is to construct an algorithm $\mathcal{B}$ for computing e'th roots modulo n.

a. Your algorithm $\mathcal{B}$ takes random n, u as input and should output $u^{1/e}$. First, show how to use $\mathcal{A}$ to construct $a \in \mathbb{Z}_n$ and $b \in \mathbb{Z}$ such that $a^e = u^b$ and $0 \neq |b| < e$.

b. Clearly $a^{1/b}$ is an e'th root of u (since $(a^{1/b})^e = u$), but unfortunately for $\mathcal{B}$, it cannot compute roots in $\mathbb{Z}_n$. Nevertheless, show how $\mathcal{B}$ can compute the eth root of u from a, u, e, b. This will complete your description of algorithm $\mathcal{B}$.
Hint: since e is prime and $0 \neq |b| < e$ we know that b and e are relatively prime. Hence, there are integers s, t so that $bs + et = 1$. Use a, u, s, t to find the e'th root of u in $\mathbb{Z}_n$.

c. Show that if we extend the domain of the function to $\mathbb{Z}_n^* \times \{0, \ldots, e\}$ then the function is no longer collision resistant.

d. Show that if the factorization of n becomes public, then the function in (1) is not even a one-way function.

Problem 4. A bad choice of primes for RSA. Let's see why when choosing an RSA modulus $n = pq$ it is important to choose the two primes p and q *independently* at random. Suppose n is generated by choosing the prime p at random, and then choosing the prime q dependent on p. In particular, suppose that p and q are close, namely $|p - q| < n^{1/4}$. Let's show that the resulting n can be easily factored.

a. Let $A = (p + q)/2$ be the arithmetic mean of p and q. Recall that $\sqrt{n}$ is the geometric mean of p and q. Show that when $|p - q| < n^{1/4}$ we have that

$$A - \sqrt{n} < 1.$$

Hint: one way to prove this is by multiplying both sides by $A + \sqrt{n}$ and then using the fact that $A \geq \sqrt{n}$ by the AGM inequality.

b. Because p and q are odd primes, we know that A is an integer. Then by part (a) we can deduce that $A = \lceil \sqrt{n} \rceil$, and therefore it is easy to calculate A from n. Show that using A and n it is easy to factor n.

Problem 5. Oblivious PRF. Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. Let $H : \mathcal{M} \to \mathbb{G}$ be a hash function. Let F be the PRF defined over $(\mathbb{Z}_q, \mathcal{M}, \mathbb{G})$ as follows:

$$F(k, m) := H(m)^k \quad \text{for } k \in \mathbb{Z}_q, m \in \mathcal{M}.$$

It is not difficult to show that this F is a secure PRF assuming the Decision Diffie-Hellman (DDH) assumption holds in the group $\mathbb{G}$ and, the hash function H is modeled as a random oracle.

Show that this PRF F can be evaluated *obliviously*. That is, show that if Bob has the key k and Alice has an input m, there is a simple protocol that allows Alice to learn $F(k, m)$ without learning anything else about k. Moreover, Bob learns nothing about m. You may assume that g and g^k are publicly known values. An oblivious PRF like this is quite handy for many applications.

a. To start the protocol, Alice generates a random $r \xleftarrow{R} \mathbb{Z}_q$ and sends to Bob $u := H(m) \cdot g^r$. Show that this u is uniformly distributed in $\mathbb{G}$ and is independent of m, so that Bob learns nothing about m.

b. Show how Bob can respond to enable Alice to learn $F(k, m)$ and nothing else.

Problem 6. In this problem we explore a vulnerability in RSA-PKCS1 v1.5 signatures that illustrates the fragility of the scheme. Let $(N, 3)$ be an RSA public-key: N is the RSA modulus and the signature verification exponent is 3. Recall that when signing a message m using PKCS1 v1.5 one first forms the block

$B =$

01	0xFF $\ldots$ 0xFF	0x00	ASN1	hash

where hash = SHA256(m). The fields are:

- 01 is a two bytes (16 bits) field set to the value 01 (for PKCS1 mode 1),
- 0xFF $\ldots$ 0xFF is a variable length padding block where each byte is set to 0xFF (i.e. the number 255),
- the $0x00$ field is 1 byte (8 bits) set to 0 indicating the end of the padding block,
- The ASN1 field encodes the type of hash function used to hash the message. For SHA256 this field holds a fixed 15 byte value.
- hash is the hash of the message m: for SHA256 this field is 32 bytes (256 bits).

The purpose of the variable length padding block is to ensure that B is about the size of N. In our case B will be padded to 256 bytes (2048 bits). Note that the ASN1 field was omitted in the lecture for simplicity.

When signing the message m the signer constructs B and then outputs $(B^{1/3} \bmod N)$ as the signature σ. Recall that the signer computes the cube root of B using his secret RSA signing key.

To verify a message/signature pair (m, σ) using the public-key $(N, 3)$ one would naively carry out the following steps:

(a) set $B \leftarrow \sigma^3 \bmod N$

(b) parse B from left to right and do:

  i. if the top most 2 bytes are not 01 reject

  ii. skip over all 0xFF bytes until reaching a 0x00 byte and skip over it too

  iii. if the next 15 bytes are not the ASN1 identifier for SHA256 reject

  iv. read the following 32 bytes (256 bits) and compare them to SHA256(m). Reject if not equal.

(c) if all the checks above pass, accept the signature

While this procedure appears to correctly verify the signature it ignores one very important step: it does not check that B contains nothing right of the hash. In particular, this procedure will accept a 256 bytes (2048 bits) block B that looks as follows:

$B^* =$

01	0xFF ... 0xFF	0x00	ASN1	hash	more bits J

where J is chosen arbitrarily by the attacker. Here the attacker shortened the variable length block of 0xFF to make room for the value J so that the total length of B^* is still 256 bytes (2048 bits).

Your goal is to show that this leads to a complete break of the signature scheme. In particular, show that just given the public-key $(N, 3)$, an attacker can forge the signature σ on any message m of its choice.

Hint: To forge the signature on some message m, first compute SHA256(m) and then construct the block B (without your appended J) so that the length of B is less than 1/3 the length of the modulus N. Say B is only 80 bytes (640 bits). To do so, simply make the variable length padding block sufficiently short.

Next, your goal is to construct a 256-byte (2048 bits) integer B^* such that:
(1) the first 80 bytes of B^* are equal to B (the remaining bits of B^* are arbitrary), and
(2) B^* is a perfect cube (i.e. is the cube of some smaller integer).
Since B^* is a perfect cube you can easily compute its real cube root σ. Then $B^* = \sigma^3$ holds over the integers and therefore the same also holds modulo N. Since the first 80 bytes of σ^3 are equal to B the signature σ will be accepted as a valid signature on m.

Show how to construct the required 256-byte B^*: it must be a perfect cube and its top 80 bytes must be equal to B. Explain how to construct this B^* and prove that your construction produces a B^* with the required properties.

History: This vulnerability was discovered by Daniel Bleichenbacher in 2006. In 2014 it was discovered that all earlier versions of Mozilla's crypto library, NSS, were vulnerable to a variant of this attack.