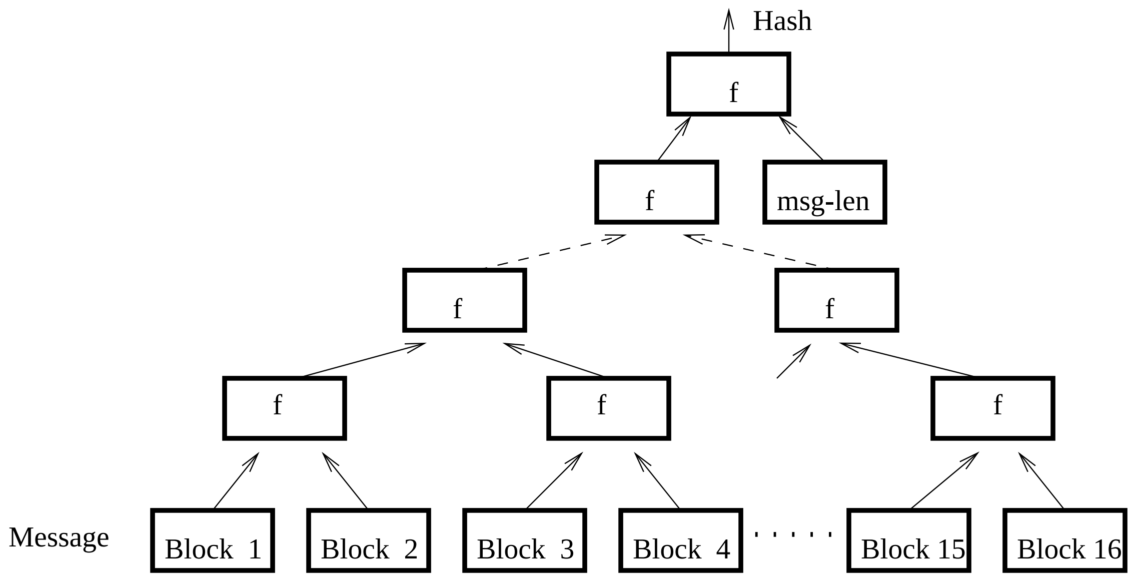
CS255: Cryptography and Computer Security **Winter 2010**

Assignment #2

Due: Wednesday, Feb. 17, 2010.

Problem 1 Merkle hash trees.

Merkle suggested a parallelizable method for constructing hash functions out of compression functions. Let f be a compression function that takes two 512 bit blocks and outputs one 512 bit block. To hash a message M one uses the following tree construction:



Prove that if one can find a collision for the resulting hash function then one can find collisions for the compression function.

Problem 2 In the lecture we saw that Davies-Meyer is often used to convert an ideal block cipher into a collision resistant compression function. Let $E(k, m)$ be a block cipher where the message space is the same as the key space (e.g. 128-bit AES). Show that the following methods do not work:

$$f_1(x, y) = E(y, x) \oplus y \quad \text{and} \quad f_2(x, y) = E(x, x) \oplus y$$

That is, show an efficient algorithm for constructing collisions for f_1 and f_2. Recall that the block cipher E and the corresponding decryption algorithm D are both known to you.

Problem 3 Collision resistance from discrete log. Let G be a finite cyclic group of prime oreder q. Let g be a generator of G.

a. Suppose that $u = g^{\alpha}$ for some x. Consider the following compression function $h : \mathbb{Z}_q^2 \to G$ defined by $h(x, y) = g^x u^y$. Show that *any* collision on h enables an attacker to compute the discrete log α.

b. Let $g_1, \ldots, g_n$ be elements in G and consider the hash function $H : \mathbb{Z}_q^n \to G$ defined by

$$H(x_1, \ldots, x_n) = g_1^{x_1} \cdots g_n^{x_n}$$

Show that an algorithm $\mathcal{A}$ that finds collisions on any such hash function H (i.e. $\mathcal{A}$ finds collisions for any choice of $g_1, \ldots, g_n$) can be used to solve discrete-log in G.

c. If we want a hash function the can hash long messages, which is more efficient: the hash function from part (b) or using Merkle-Damgard on the hash function from part (a) ?

Problem 4 Suppose user A is broadcasting packets to n recipients $B_1, \ldots, B_n$. Privacy is not important but integrity is. In other words, each of $B_1, \ldots, B_n$ should be assured that the packets he is receiving were sent by A. User A decides to use a MAC.

a. Suppose user A and $B_1, \ldots, B_n$ all share a secret key k. User A MACs every packet she sends using k. Each user B_i can then verify the MAC. Using at most two sentences explain why this scheme is insecure, namely, show that user B_1 is not assured that packets he is receiving are from A.

b. Suppose user A has a set $S = \{k_1, \ldots, k_m\}$ of m secret keys. Each user B_i has some subset $S_i \subseteq S$ of the keys. When A transmits a packet she appends m MACs to it by MACing the packet with each of her m keys. When user B_i receives a packet he accepts it as valid only if all MAC's corresponding to keys in S_i are valid. What property should the sets $S_1, \ldots, S_n$ satisfy so that the attack from part (a) does not apply? We are assuming all users $B_1, \ldots, B_n$ are sufficiently far apart so that they cannot collude.

c. Show that when $n = 10$ (i.e. ten recipients) the broadcaster A need only append 5 MAC's to every packet to satisfy the condition of part (b). Describe the sets $S_1, \ldots, S_{10} \subseteq \{k_1, \ldots, k_5\}$ you would use.

Problem 5 Strengthening hashes and MACs.

a. Suppose we are given two hash functions $H_1, H_2 : \{0,1\}^* \to \{0,1\}^n$ (for example SHA1 and MD5) and are told that both hash functions are collision resistant. We, however, do not quite trust these claims. Our goal is to build a hash function $H_{12} : \{0,1\}^* \to \{0,1\}^m$ that is collision resistant assuming *at least* one of H_1, H_2 are collision resistant. Give the best construction you can for H_{12} and prove that a collision finder for your H_{12} can be used to find collisions for both H_1 and H_2

(this will prove collision resistance of H_{12} assuming one of H_1 or H_2 is collision resistant). Note that a straight forward construction for H_{12} is fine, as long as you prove security in the sense above.

b. Same questions as part (a) for Message Authentication Codes (MACs). Prove that an existential forger under a chosen message attack on your MAC_{12} gives an existential forger under a chosen message attack for both MAC_1 and MAC_2. Again, a straight forward construction is acceptable, as long as you prove security. The proof of security here is a bit more involved than in part (a). Make sure your proof defines explicitely how the MAC_1 forger works given the MAC_{12} forger.

Problem 6 In this problem, we see why it is a really bad idea to choose a prime $p = 2^k + 1$ for discrete-log based protocols: the discrete logarithm can be efficiently computed for such p. Let g be a generator of $\mathbb{Z}_p^*$.

a. Show how one can compute the least significant bit of the discrete log. That is, given $y = g^x$ (with x unknown), show how to determine whether x is even or odd by computing $y^{(p-1)/2} \bmod p$.

b. If x is even, show how to compute the 2nd least significant bit of x.
Hint: consider $y^{(p-1)/4} \bmod p$.

c. Generalize part (b) and show how to compute all of x.
Hint: let $b \in \{0, 1\}$ be the LSB of x obtained using part (a). Try setting $y_1 \leftarrow y/g^b$ and observe that y_1 is an even power of g. Then use part (b) to deduce the second least significant bit of x. Show how to iterate this procedure to recover all of x.

d. Briefly explain why your algorithm does not work for a random prime p.