The RSA Trapdoor Permutation

Dan Boneh
Stanford University

Review: arithmetic mod composites

- Let $N = p \cdot q$ where p,q are prime
- Notation: $Z_N = \{0,1,2,\ldots,N-1\}$

  $(Z_N)^* = \{\text{invertible elements in } Z_N\}$
- Facts:
  - $x \in Z_N$ is in $(Z_N)^*$ $\Leftrightarrow$ $\gcd(x,N) = 1$
  - Number of elements in $(Z_N)^*$ is $\varphi(N) = (p-1)(q-1)$
- Euler's thm: $\forall\, x \in (Z_N)^* \;:\; x^{\varphi(N)} = 1$

The RSA trapdoor permutation

- First published:
  - Scientific American, Aug. 1977.

    (after some censorship entanglements)
- Currently the "work horse" of Internet security:
  - Most Public Key Infrastructure (PKI) products.
  - SSL/TLS: Certificates and key-exchange.
  - Secure e-mail: PGP, Outlook, …

The RSA trapdoor permutation

- Parameters: $N=pq$. $N \approx 1024$ bits. $p,q \approx 512$ bits.
  e – encryption exponent. $\gcd(e, \varphi(N)) = 1$.

- 1-to-1 function: $\mathbf{RSA(M) = M^e} \in Z_N^*$ where $M \in Z_N^*$

- Trapdoor: **d** – decryption exponent.
  Where $e \cdot d = 1 \pmod{\varphi(N)}$

- Inversion: $\mathbf{RSA(M)^d} = M^{ed} = M^{k\varphi(N)+1} = (M^{\varphi(N)})^k \cdot \mathbf{M} = M$

- (n,e,t,ε)-RSA Assumption: For all t-time algs. A:

$$\Pr\left[\ A(N,e,x) = x^{1/e}\ (N)\ :\ \begin{array}{c} p,q \xleftarrow{R} \text{n-bit primes}, \\ N \leftarrow pq,\quad x \xleftarrow{R} Z_N^* \end{array}\ \right] < \varepsilon$$

Textbook RSA enc. is insecure

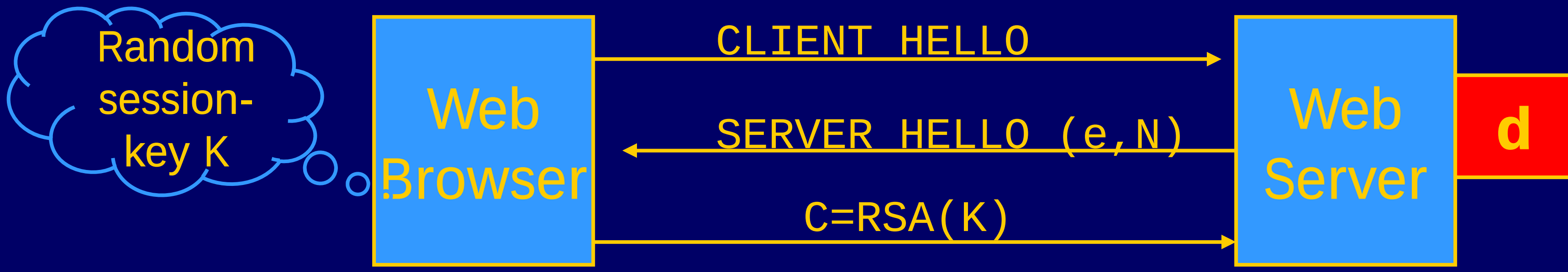
- Textbook RSA encryption:
  - public key: **(N,e)** Encrypt: $C = M^e \pmod N$
  - private key: **d** Decrypt: $C^d = M \pmod N$

    $(M \in Z_N^*)$

- Completely insecure cryptosystem:
  - Does not satisfy basic definitions of security.
  - Many attacks exist.

- The RSA trapdoor permutation is not an enc. scheme!

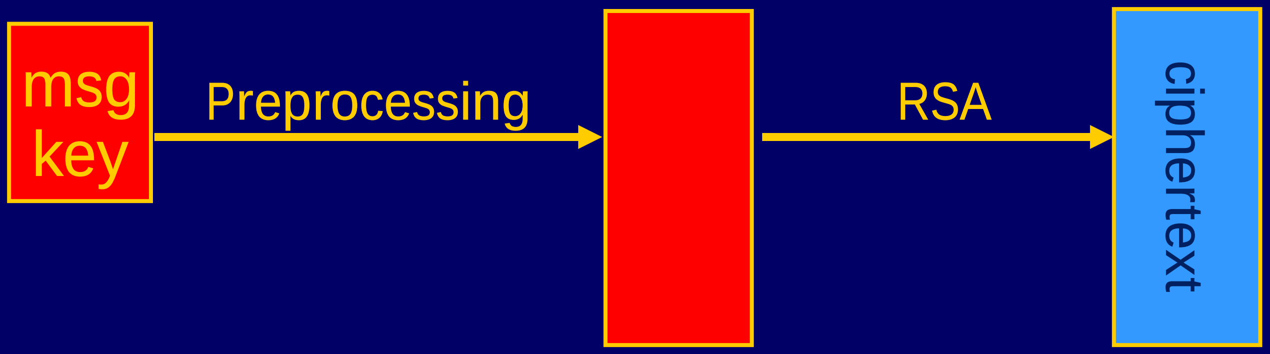
A simple attack on textbook RSA



- Session-key K is 64 bits. View $K \in \{0,\ldots,2^{64}\}$
  Eavesdropper sees: $C = K^e \pmod N$.
- Suppose $K = K_1 \cdot K_2$ where $K_1, K_2 < 2^{34}$. (prob. ≈20%)
  Then: $C/K_1^e = K_2^e \pmod N$
- Build table: $C/1^e, C/2^e, C/3^e, \ldots, C/2^{34e}$. time: 2^{34}
  For $K_2 = 0,\ldots, 2^{34}$ test if K_2^e is in table. time: $2^{34} \cdot 34$
- Attack time: $\approx 2^{40} \ll 2^{64}$

RSA enc. in practice

- Never use textbook RSA.
- RSA in practice (generic enc. with trapdoor func. is not often used) :



- Main question:
  - How should the preprocessing be done?
  - Can we argue about security of resulting system?

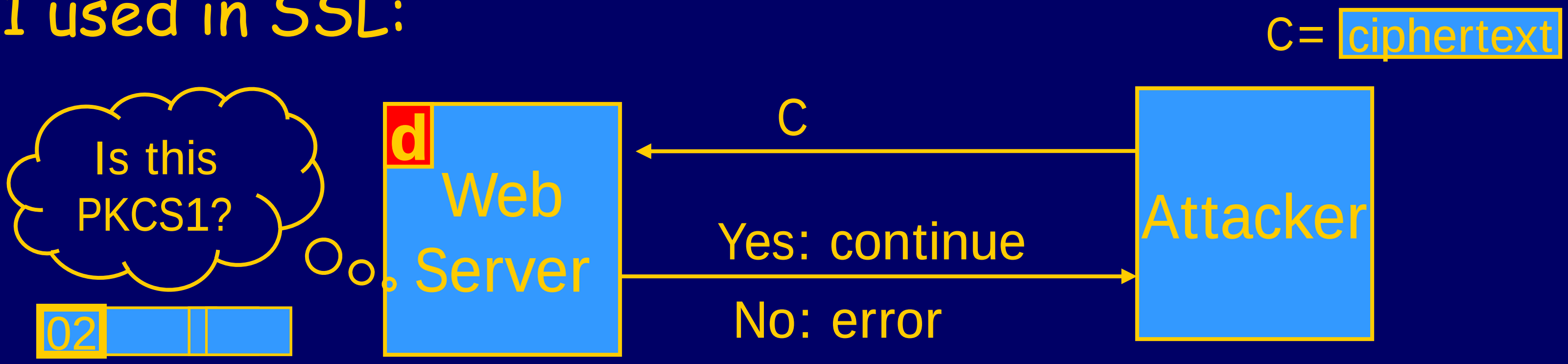
PKCS1 V1.5

- PKCS1 mode 2: (encryption)

- Resulting value is RSA encrypted.
- Widely deployed in web servers and browsers.
- No security analysis !!

Attack on PKCS1

- Bleichenbacher 98. Chosen-ciphertext attack.
- PKCS1 used in SSL:



$\Rightarrow$ attacker can test if 16 MSBs of plaintext = '02'.

- Attack: to decrypt a given ciphertext C do:
  - Pick $r \in Z_N$. Compute $C' = r^e \cdot C = (r \cdot \text{PKCS1}(M))^e$.
  - Send C' to web server and use response.

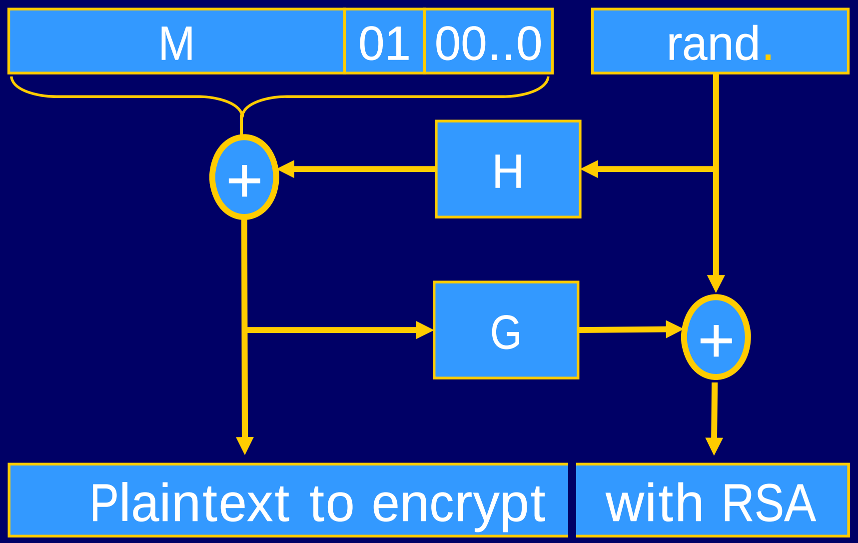
Review: chosen CT security (CCS)

- No efficient attacker can win the following game: (with non-negligible advantage)

Attacker wins if b=b'

PKCS1 V2.0 - OAEP

- New preprocessing function: OAEP [BR94]



$\in \{0,1\}^{n-1}$

Check pad
on decryption.
Reject CT if invalid.

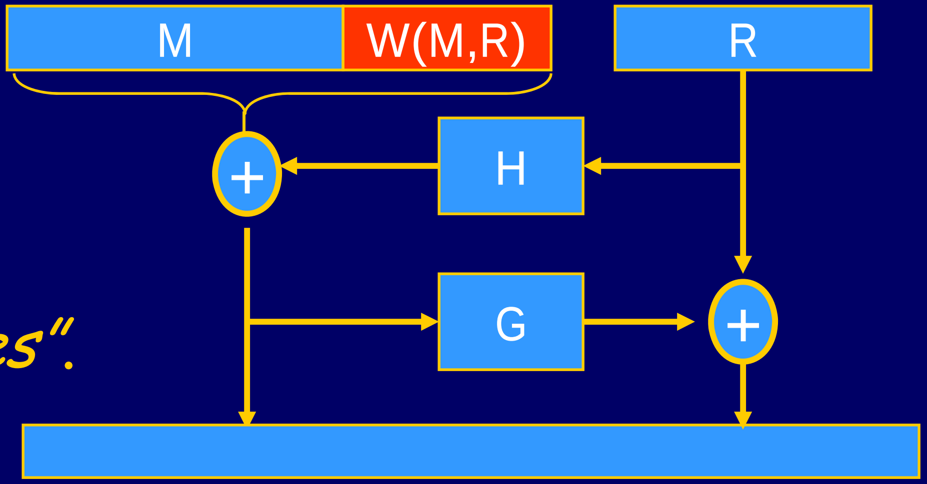
- Thm [FOPS'01] : RSA is trap-door permutation $\Rightarrow$
  RSA-OAEP is CCS when H,G are "*random oracles*".

- In practice: use SHA-256 for H and G.

OAEP Improvements

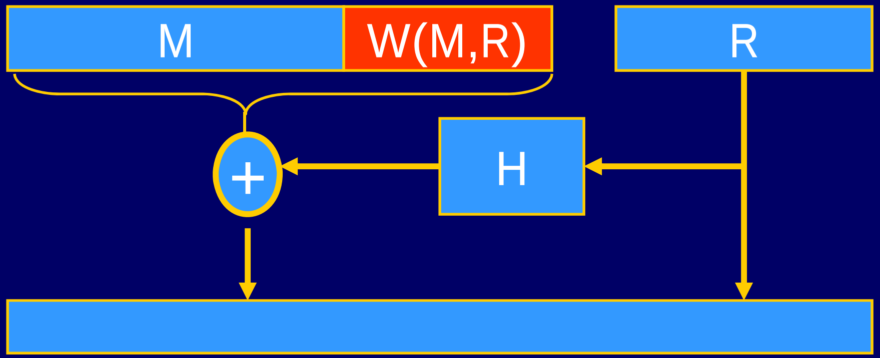
- OAEP+: [Shoup'01]

  ∀ trap-door permutation F
  F-OAEP+ is CCS when
  H,G,W are "*random oracles*".



- SAEP+: [B'01]

  RSA trap-door perm ⇒
  RSA-SAEP+ is CCS when
  H,W are "*random oracle*".



Subtleties in implementing OAEP [M '00]

```
OAEP-decrypt(C) {
    error = 0;
       ........
    if ( RSA^-1(C) > 2^(n-1) )
         { error =1;  goto exit; }
       ........
    if ( pad(OAEP^-1(RSA^-1(C))) != "01000" )
         { error = 1;  goto exit; }
}
```

- Problem: timing information leaks type of error.
  $\Rightarrow$ Attacker can decrypt any ciphertext C.
- Lesson: Don't implement RSA-OAEP yourself …

Part II:
Is RSA a One-Way Function?

Is RSA a one-way function?

- To invert the RSA one-way function (without d) attacker must compute:

  M from $C = M^e \pmod{N}$.

- How hard is computing e'th roots modulo N ??

- Best known algorithm:
  - Step 1: factor N. (hard)
  - Step 2: Find e'th roots modulo p and q. (easy)

Shortcuts?

- Must one factor N in order to compute e'th roots? Exists shortcut for breaking RSA without factoring?

- To prove no shortcut exists show a reduction:
  - Efficient algorithm for e'th roots mod N

    $\Rightarrow$ efficient algorithm for factoring N.
  - Oldest problem in public key cryptography.

- Evidence no reduction exists: (BV'98)
  - "Algebraic" reduction $\Rightarrow$ factoring is easy.
  - Unlike Diffie-Hellman (Maurer'94).

Improving RSA's performance

- To speed up RSA decryption use small private key d. $C^d = M \pmod{N}$

  - Wiener87: if $d < N^{0.25}$ then RSA is insecure.
  - BD'98: if $d < N^{0.292}$ then RSA is insecure (open: $d < N^{0.5}$)
  - Insecure: priv. key d can be found from (N,e).
  - Small d should never be used.

Wiener's attack

- Recall: $e \cdot d = 1 \pmod{\varphi(N)}$

  $\Rightarrow \quad \exists\, k \in Z: \quad e \cdot d = k \cdot \varphi(N) + 1$

  $$\Rightarrow \left| \frac{e}{\varphi(N)} - \frac{k}{d} \right| \leq \frac{1}{d\varphi(N)}$$

$\varphi(N) = N-p-q+1 \quad \Rightarrow \quad |N - \varphi(N)| \leq p+q \leq 3\sqrt{N}$

$$d \leq N^{0.25}/3 \quad \Rightarrow \left| \frac{e}{N} - \frac{k}{d} \right| \leq \frac{1}{2d^2}$$

Continued fraction expansion of e/N gives k/d.

$e \cdot d = 1 \pmod{k} \quad \Rightarrow \quad \gcd(d,k)=1$

RSA With Low public exponent

- To speed up RSA encryption (and sig. verify) use a small e. $C = M^e \pmod{N}$
- Minimal value: $e=3$ ($\gcd(e, \varphi(N)) = 1$)
- Recommended value: $e=65537=2^{16}+1$

  Encryption: 17 mod. multiplies.
- Several weak attacks. Non known on RSA-OAEP.
- Asymmetry of RSA: fast enc. / slow dec.
  - ElGamal: approx. same time for both.

Implementation attacks

- Attack the implementation of RSA.
- Timing attack: (Kocher 97)
  The time it takes to compute C^d (mod N) can expose d.
- Power attack: (Kocher 99)
  The power consumption of a smartcard while it is computing C^d (mod N) can expose d.
- Faults attack: (BDL 97)
  A computer error during C^d (mod N) can expose d.

  OpenSSL defense: check output. 5% slowdown.

Key lengths

- Security of public key system should be comparable to security of block cipher.

NIST:

Cipher key-size	Modulus size
$\leq$ 64 bits	512 bits.
80 bits	1024 bits
128 bits	3072 bits.
256 bits (AES)	**15360** bits

- High security $\Rightarrow$ very large moduli.

  Not necessary with Elliptic Curve Cryptography.